Ceng 124
Discrete Structures

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Topics

- 2.1 Sets Theory
- 2.2 Sets Operations

2.1 Sets Theory

Many important discrete structures are built using sets, which are collections of objects.

sets->relations->graphs->finite state machines-> model computing machines

A *set* is an unordered collection of objects, called *elements* or *members* of the set. A set is said to *contain* its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

A = {a1, a2, …, an} “A contains…”

Order of elements is meaningless

It does not matter how often the same element is listed.

Set Equality

- Sets A and B are equal if and only if they contain exactly the same elements.
- Examples:

  $A = \{9, 2, 7, -3\}$, $B = \{7, 9, -3, 2\}$: $A = B$

  $A = \{dog, cat, horse\}$, $B = \{cat, horse, squirrel, dog\}$: $A \neq B$

  $A = \{dog, cat, horse\}$, $B = \{cat, horse, dog, dog\}$: $A = B$

Roster Method

- There are several ways to describe a set. One way is to list all the members of a set, when this is possible. We use a notation where all members of the set are listed between braces. This way of describing a set is known as the **roster method.**
- The set V of all vowels in the English alphabet can be written as

  $V = \{a, e, i, o, u\}$.
- The set O of odd positive integers less than 10 can be expressed by

  $O = \{1, 3, 5, 7, 9\}$.

Set Builder Notation

- Another way to describe a set is to use set builder notation:

  $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$,

  $O = \{x \in \mathbf{Z}+ \mid x \text{ is odd and } x < 10\}$.

Standard Sets

- Natural numbers **N** = {0, 1, 2, 3, …}
- Integers **Z** = {…, -2, -1, 0, 1, 2, …}
- Positive Integers **Z+** = {1, 2, 3, 4, …}
- Real Numbers **R** = {47.3, -12, π, …}
- Rational Numbers **Q** = { p/q | p ∈ Z, q ∈ Z, and q !=0 }

  Q = {2/5, 13/5, 15, …}

Equal Sets

Two sets are *equal* if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

- The sets {1, 3, 5} and {3, 5, 1} are equal, because they have the same elements.
- Also {1, 3, 3, 3, 5, 5, 5, 5} is the same as the set {1, 3, 5} because they have the same elements.

Subsets

The set A is a *subset* of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

- We can completely formalize this:

$$A \subseteq B \Leftrightarrow \forall x\ (x \in A \rightarrow x \in B)$$

- Examples:

A = {3, 9}, B = {5, 9, 1, 3},	$A \subseteq B$?	True
A = {3, 3, 3, 9}, B = {5, 9, 1, 3},	$A \subseteq B$?	True
A = {1, 2, 3}, B = {2, 3, 4},	$A \subseteq B$?	False

Empty Set and Singleton Set

- **Empty Set:** The ***void* set**, the ***null* set**, the ***empty* set**, denoted ∅, is the set with no members. Therefore, ∅ is a subset of every set.
- The empty set can also be denoted by { }.
- A set with one element is called a **singleton set**.
- A common error is to confuse the empty {∅} has one more element than ∅. set ∅ with the set {∅}, which is a singleton set. The single element of the set {∅} is the empty set itself!

Size of the Set

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a *finite set* and that n is the *cardinality* of S. The cardinality of S is denoted by $|S|$.

- Let S be the set of letters in the English alphabet. Then $|S| = 26$.
- Because the null set has no elements, it follows that $|\emptyset| = 0$.

A set is said to be *infinite* if it is not finite.

Cardinality of Sets

- If a set S contains n distinct elements, $n \in \mathbf{N}$, we call S a finite set with cardinality n.
- Examples:

$A = \{M, B, P\}$,	$\|A\| = 3$
$B = \{1, \{2, 3\}, \{4, 5\}, 6\}$,	$\|B\| = 4$
$C = \varnothing$	$\|C\| = 0$
$D = \{ x \in \mathbf{N} \mid x \leq 7000 \}$	$\|D\| = 7001$
$E = \{ x \in \mathbf{N} \mid x \geq 7000 \}$	E is Infinite!

Power Sets

Given a set S, the *power set* of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

- What is the power set of the set {0, 1, 2}?
- The power set *P*({0, 1, 2}) is the set of all subsets of {0, 1, 2}. Hence,

  P({0, 1, 2}) = {∅, {0}, {1}, {2}, {0, 1}, {0, 2}, {1, 2}, {0, 1, 2}}.

Power Set (cont.)

- 2^A or P(A) "power set of A"

  $2^A = \{B \mid B \subseteq A\}$ (contains all subsets of A)

- Examples:

  $A = \{x, y, z\}$

  $2^A = \{\varnothing, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$

  $A = \varnothing$

  $2^A = \{\varnothing\}$

  Note: $|A| = 0$, $|2^A| = 1$

Cartesian Products

Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

Example: A = {x, y}, B = {a, b, c}

$A \times B$ = {(x, a), (x, b), (x, c), (y, a), (y, b), (y, c)}

Set Operations

- Union: $A \cup B = \{x \mid x \in A \vee x \in B\}$

  Example: $A = \{a, b\}$, $B = \{b, c, d\}$ $A \cup B = \{a, b, c, d\}$

- Intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$

  Example: $A = \{a, b\}$, $B = \{b, c, d\}$ $A \cap B = \{b\}$

- Disjoint: if their (A, B) intersection is empty

  Example Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Because $A \cap B = \emptyset$,

  A and B are disjoint.

- Difference: Let A and B be sets. The difference of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B.

  Example The difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{5\}$.

Set Identities

TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws

$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Proof of De Morgan's Law

- Use set builder notation and logical equivalences to establish (prove) the first De Morgan law $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} \\
&= \{x \mid \neg(x \in (A \cap B))\} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} \\
&= \{x \mid x \notin A \vee x \notin B\} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} \\
&= \overline{A} \cup \overline{B}
\end{aligned}
$$

Computer Representation of Sets

- Example: Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, and the ordering of elements of U has the elements in increasing order; that is, ai = i. What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?
- all odd integers in U, namely, {1, 3, 5, 7, 9}, by the string

  10 1010 1010.
- all even integers in U, namely, {2, 4, 6, 8, 10}, by the string

  01 0101 0101.
- do not exceed 5, namely, {1, 2, 3, 4, 5}, by the string

  11 1110 0000.

Computer Representation of Sets (cont.)

- Example 2 : The bit strings for the sets {1, 2, 3, 4, 5} and {1, 3, 5, 7, 9} are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the union and intersection of these sets.
- The bit string for the union of these sets is

  11 1110 0000 ∨ 10 1010 1010 = 11 1110 1010,

  which corresponds to the set {1, 2, 3, 4, 5, 7, 9}.

  The bit string for the intersection of these sets is

- 11 1110 0000 ∧ 10 1010 1010 = 10 1010 0000,

  which corresponds to the set {1, 3, 5}.