Ceng 124
Discrete Structures

2018-2019 Spring Semester

by Sibel T. Özyer, Spring 2019

Topics

- 9.3 Representing Relations
  - using Matrices
  - using Directed Graphs
- 9.5 Equivalence Relations

Introduction

- First way is to list the ordered pairs
- Second way is through matrices
- Third way is through direct graphs

Representing Relations using Matrices

- Suppose that R is a relation from $A = \{a1, a2, . . . , am\}$ to $B = \{b1, b2, . . . , bn\}$.
- The relation R can be represented by the matrix $\mathbf{M}_R=[mij]$, where

$$m_{ij} = \begin{cases} 1 & if\ (a_i, b_j) \in R \\ 0 & otherwise \end{cases}$$

- Generally, matrices are appropriate for the representation of relations in computer programs.

Example1

- Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

- Solution: Because $R = \{(2, 1), (3, 1), (3, 2)\}$, the matrix for R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

The 1s in M_R show that the pairs (2, 1), (3, 1), and (3, 2) belong to R. The 0s show that no other pairs belong to R.

Example2

- Let $A = \{a1, a2, a3\}$ and $B = \{b1, b2, b3, b4, b5\}$. Which ordered pairs are in the relation R represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

- Because R consists of those ordered pairs (ai, bj) with $mij = 1$, it follows that
- $R = \{(a1, b2), (a2, b1), (a2, b3), (a2, b4), (a3, b1), (a3, b3), (a3, b5)\}$.

Reflexive Relation on Zero-One Matrix

- A relation *R* on *A* is reflexive if *(a, a)* ∈ *R* whenever *a* ∈ *A*. Thus, *R* is reflexive if and only if *(ai, ai)* ∈ *R* for *i* = 1, 2, . . . , *n*. Hence, *R* is reflexive if and only if *mii* = 1, for *i* = 1, 2, . . . , *n*. In other words, *R* is reflexive if all the elements on the main diagonal of *MR* are equal to 1.

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

Figure 1: Zero-One Matrix for a Reflexive Relation

Symmetric Relation on Zero-One Matrix

- R is symmetric if and only if $m_{ji} = 1$ whenever $m_{ij} = 1$. This also means $m_{ji} = 0$ whenever $m_{ij} = 0$. Consequently, R is symmetric if and only if $m_{ij} = m_{ji}$, for all pairs of integers i and j with $i = 1, 2, . . . , n$ and $j = 1, 2, . . . , n$.
- Also the definition of the transpose of a matrix, we see that R is symmetric if and only if $M_R = (M_R)^t$.

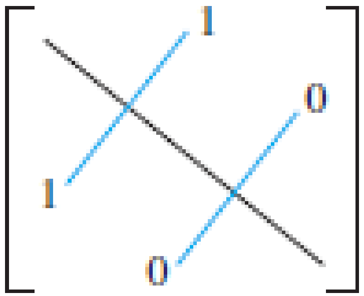


Figure 2: Zero-One Matrix for a Symmetric Relation

Antisymmetric Relation on Zero-One Matrix

- The relation R is antisymmetric if and only if $(a, b) \in R$ and $(b, a) \in R$ imply that $a = b$. Consequently, the matrix of an antisymmetric relation has the property that if $mij = 1$ with $i = j$, then $mji = 0$. In other words, either $mij = 0$ or $mji = 0$ when $i = j$. The form of the matrix for an antisymmetric relation is illustrated in Figure 3.

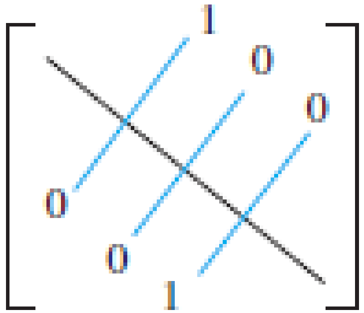


Figure 3: Zero-One Matrix for a Antisymmetric Relation

Example1

- Suppose that the relation *R* on a set is represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is *R* reflexive, symmetric, and/or antisymmetric?

Solution

- Because all the diagonal elements of this matrix are equal to 1, *R* is reflexive.
- Moreover, because ***M*R** is symmetric, it follows that *R* is symmetric.
- It is also easy to see that *R* is not antisymmetric.

Representing Relations using Directed Graphs (Digraph)

A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a, b), and the vertex b is called the *terminal vertex* of this edge.

An edge of the form *(a, a)* is represented using an arc from the vertex *a* back to itself. Such an edge is called a *loop.*

Sample of Directed Graphs

- The directed graph with vertices *a*, *b*, *c*, and *d*, and edges *(a, b)*, *(a, d)*, *(b, b)*, *(b, d)*, *(c, a)*, *(c, b)*, and *(d, b)* is displayed in Figure 4.

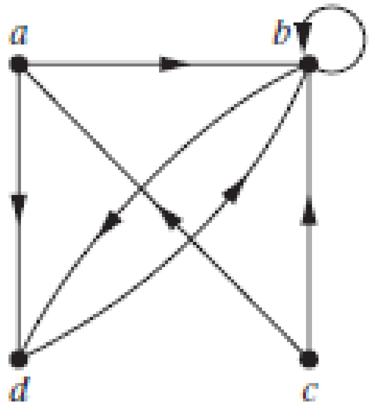


Figure 4: A directed graph

Directed graphs give a visual display of information about relations.

Sample of Directed Graphs (cont.)

- The directed graph of the relation

  $R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$

  on the set {1, 2, 3, 4} is shown in Figure 5.

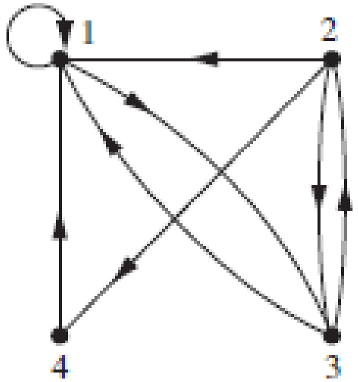


Figure 5: A directed graph

Question1

- What are the ordered pairs in the relation *R* represented by the directed graph shown in Figure 6?
- Solution: The ordered pairs (x, y) in the relation are

  $R = \{(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)\}$.

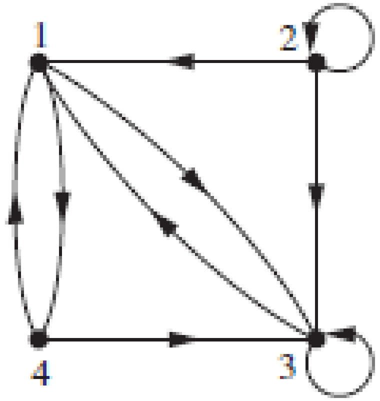
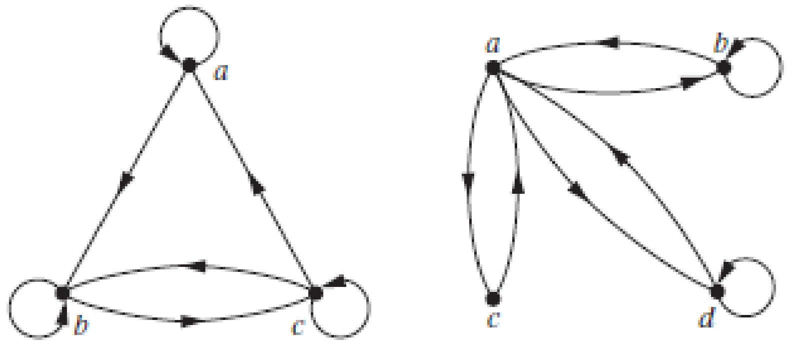


Figure 6: A directed graph

- Each of these pairs corresponds to an edge of the directed graph, with $(2, 2)$ and $(3, 3)$ corresponding to loops.

Question2

- Determine whether the relations for the directed graphs shown in Figure 7 are reflexive, symmetric, antisymmetric, and/or transitive.



(a) Directed graph of *R* (b) Directed graph of *S*

Figure 7: The directed graph of the Relations R and S

Solutions of Question2

- Solution (a):

  Reflexive: loops at every vertex of the directed graph of *R*.

  Not Symmetric: an edge from a to b (a,b) but not b to a (b,a).

  Not Antisymmetric: There are edges in both directions connecting *b* and *c*. (b,c) and (c,b).

  Not Transitive: There is an edge from *a* to *b* and an edge from *b* to *c*, but no edge from *a* to *c*.

- Solution (b):

  Not Reflexive: loops are not present at all the vertices of the directed graph of *S*.

  Symmetric and Not Antisymmetric: every edge between distinct vertices is accompanied by an edge in the opposite direction.

  Not Transitive: because *(c, a)* and *(a, b)* belong to *S*, but *(c, b)* does not belong to *S*.

9.5 Equivalence Relations

A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Question1

- Let R be the relation on the set of real numbers such that aRb if and only if $a - b$ is an integer. Is R an equivalence relation?
- *Solution:*
- Because $a - a = 0$ is an integer for all real numbers a, aRa for all real numbers a. Hence, R is reflexive.
- Now suppose that aRb. Then $a - b$ is an integer, so $b - a$ is also an integer. Hence, bRa. It follows that R is symmetric.
- If aRb and bRc, then $a - b$ and $b - c$ are integers.

  Therefore, $a - c = (a - b) + (b - c)$ is also an integer.

  Hence, aRc. Thus, R is transitive.
- Consequently, R is an equivalence relation.

Question2

- Let $n = 3$ and let S be the set of all bit strings. Then sR_3t either when $s = t$ or both s and t are bit strings of length 3 or more that begin with the same three bits. For instance, $01R_301$ and $00111R_300101$, but $01\not R_3010$ and $01011\not R_301110$.
- Show that for every set S of strings and every positive integer n, R_n is an equivalence relation on S.
- Solution: The relation R_n is reflexive because $s = s$, so that sR_ns. If sR_nt , and tR_ns. We conclude that R_n is symmetric. Now suppose that sR_nt and tR_nu. Then $s = t$, $t = u$ deduce that either $s = u$ or both s and u are n characters long and s and u begin with the same n characters. Consequently, R_n is transitive.
- It follows that R_n is an equivalence relation.