Ceng 124
Discrete Structures

2018-2019 Spring Semester

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Topics

- 9.6 Partial Ordering
  - Definition
  - Arbitrary Poset
  - Total Ordering
  - Lexicographic Order
  - Hasse Diagram
  - Upper and Lower Bounds
  - Lattices

Introduction

- We often use relations to order some or all of the elements of sets.
- words using the relation containing pairs of words (x, y), where x comes before y in the dictionary.
- schedule projects using the relation consisting of pairs (x, y), where x and y are tasks in a project such that x must be completed before y begins.
- the set of integers using the relation containing the pairs (x, y), where x is less than y.

Definition of Partial Order

A relation R on a set S is called a *partial ordering* or *partial order* if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

Example1

- Show that the “greater than or equal” relation (≥) is a partial ordering on the set of integers.
- *Solution:* Because $a \geq a$ for every integer a, ≥ is reflexive.
- If $a \geq b$ and $b \geq a$, then $a = b$. Hence, ≥ is antisymmetric.
- Finally, ≥ is transitive because $a \geq b$ and $b \geq c$ imply that $a \geq c$.
- It follows that ≥ is a partial ordering on the set of integers and $(\mathbf{Z}, \geq)$ is a poset.

Example 2

- The divisibility relation | is a partial ordering on the set of positive integers, because it is reflexive, antisymmetric, and transitive.We see that (**Z**+, |) is a poset. Recall that (**Z**+ denotes the set of positive integers.)
- *Solution:* Because $a \mid a$ whenever a is a positive integer, the "divides" relation is reflexive.
- This relation is not symmetric because $1|2$, but $2 \nmid 1$. It is antisymmetric, for if a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$.
- Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. It follows that this relation is transitive.
- Therefore, divisibilty relation is a partial ordering on the set of positive integers.

Arbitrary Poset (Partial Ordering Set)

- In different posets different symbols such as ≤, ⊆, and |, are used for a partial ordering.
- However, we need a symbol that we can use when we discuss the ordering relation in an arbitrary poset. Customarily, the notation $a \preccurlyeq b$ is used to denote that *(a, b)* ∈ *R* in an arbitrary poset *(S,R)*. This notation is used because the "less than or equal to" relation on the set of real numbers is the most familiar example of a partial ordering and the symbol is similar to the ≤ symbol.

Arbitrary Poset (cont.)

- When a and b are elements of the poset $(S, \preccurlyeq)$, it is not necessary that either $a \preccurlyeq b$ or $b \preccurlyeq a$. For instance, in $(P(\mathbf{Z}), \subseteq)$, {1, 2} is not related to {1, 3}, and vice versa, because neither set is contained within the other. Similarly, in $(\mathbf{Z}^+, |)$, 2 is not related to 3 and 3 is not related to 2, because $2 \nmid 3$ and $3 \nmid 2$.

The elements a and b of a poset $(S, \preccurlyeq)$ are called *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called *incomparable*.

Example

- In the poset $(\mathbf{Z}^+, |)$, are the integers 3 and 9 comparable? Are 5 and 7 comparable?
- *Solution:* The integers 3 and 9 are comparable, because $3 \mid 9$. The integers 5 and 7 are incomparable, because $5 \nmid 7$ and $7 \nmid 5$.

Total Ordering

- The adjective partial is used to describe partial orderings because pairs of elements may be incomparable. When every two elements in the set are comparable, the relation is called a **total ordering**.

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order.* A totally ordered set is also called a *chain*.

Examples

- The poset $(\mathbf{Z},\leq)$ is totally ordered, because $a \leq b$ or $b \leq a$ whenever a and b are integers.
- The poset $(\mathbf{Z}^+, \mid)$ is not totally ordered because it contains elements that are incomparable, such as 5 and 7.

Lexicographic Order

- The words in a dictionary are listed in alphabetic, or lexicographic, order, which is based on the ordering of the letters in the alphabet.
- Ordering of strings.

Question

- Determine whether $(3, 5) \prec (4, 8)$, whether $(3, 8) \prec (4, 5)$, and whether $(4, 9) \prec (4, 11)$ in the poset $(\mathbf{Z} \times \mathbf{Z},)$, where is the lexicographic ordering constructed from the usual $\leq$ relation on $\mathbf{Z}$.
- *Solution:* Because $3 < 4$, it follows that $(3, 5) \prec (4, 8)$ and that $(3, 8) \prec (4, 5)$. We have $(4, 9) \prec (4, 11)$, because the first entries of $(4, 9)$ and $(4, 11)$ are the same but $9 < 11$.

Example

- Consider the set of strings of lowercase English letters:
  - *discreet* $\prec$ *discrete*,
- because these strings differ first in the seventh position, and $e \prec t$. Also,
  - *discreet* $\prec$ *discreetness*,
- because the first eight letters agree, but the second string is longer. Furthermore,
  - *discrete* $\prec$ *discretion*,
- because

discrete $\prec$ *discreti*.

Hasse Diagram

- Many edges in the directed graph for a finite poset do not have to be shown because they must be present.
- For instance, consider the directed graph for the partial ordering $\{(a, b) \mid a \leq b\}$ on the set {1, 2, 3, 4}, shown in Figure 1.

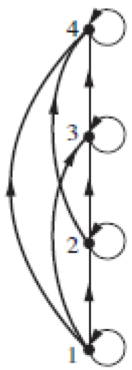


Figure 1 Constructing the Hasse Diagram for $(\{1, 2, 3, 4\},\leq)$.

Hasse Diagram (cont.)

- Because this relation is a partial ordering, it is reflexive, and its directed graph has loops at all vertices. Consequently, we do not have to show these loops because they must be present; in Figure 2 (b) loops are not shown.
- Because a partial ordering is transitive, we do not have to show those edges that must be present because of transitivity in Figure 2(c).

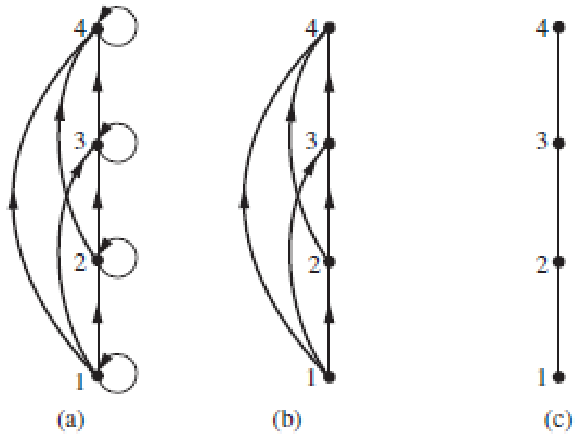


Figure 2: Constructing the Hasse Diagram for $(\{1, 2, 3, 4\}, \leq)$.

Hasse Diagram (cont.)

- If we assume that all edges are pointed “upward” (as they are drawn in the figure), we do not have to show the directions of the edges; Figure 2(c) does not show directions.

Procedure to represent Hasse Diagram

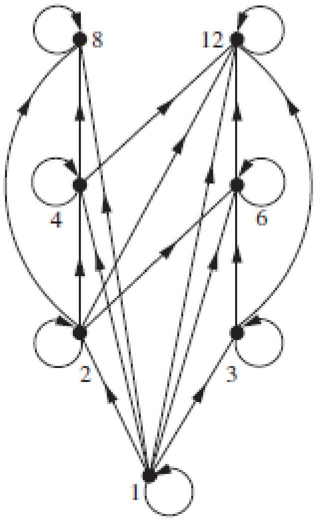
- Start with the directed graph for this relation. Because a partial ordering is reflexive, a loop (a, a) is present at every vertex a. Remove these loops.
- Next, remove all edges that must be in the partial ordering because of the presence of other edges and transitivity. That is, remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec x$.
- Finally, arrange each edge so that its initial vertex is below its terminal vertex. Remove all the arrows on the directed edges, because all edges point "upward" toward their terminal vertex.

Question

- Draw the Hasse diagram representing the partial ordering $\{(a, b) \mid a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.

Solution

- Begin with the digraph for this partial order, as shown in Figure 3(a).



(a)

Figure 3(a) Constructing the Hasse Diagram of ({1, 2, 3, 4, 6, 8, 12}, |).

Solution (cont.)

- Remove all loops, as shown in Figure 3(b).

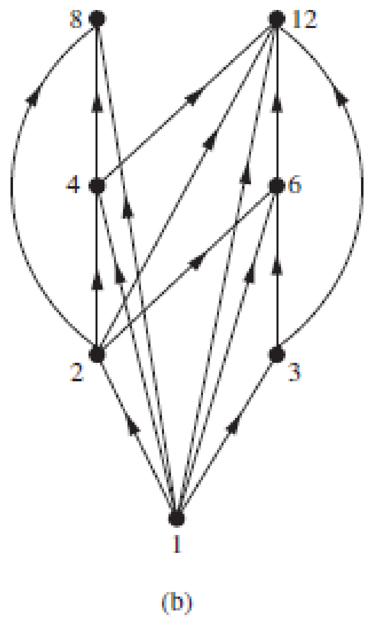


Figure 3(b) Constructing the Hasse Diagram of ({1, 2, 3, 4, 6, 8, 12}, |).

Solution (cont.)

- Then delete all the edges implied by the transitive property. These are *(*1, 4*)*, *(*1, 6*)*, *(*1, 8*)*, *(*1, 12*)*, *(*2, 8*)*, *(*2, 12*)*, and *(*3, 12*)*.
- Arrange all edges to point upward, and delete all arrows to obtain the Hasse diagram.

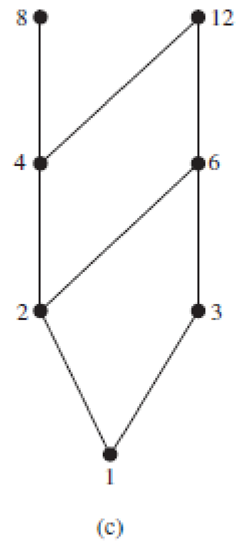


Figure 3(c) Constructing the Hasse Diagram of ({1, 2, 3, 4, 6, 8, 12}, |).

Solution (cont.)

- The resulting Hasse diagram is shown in Figure 3(c).

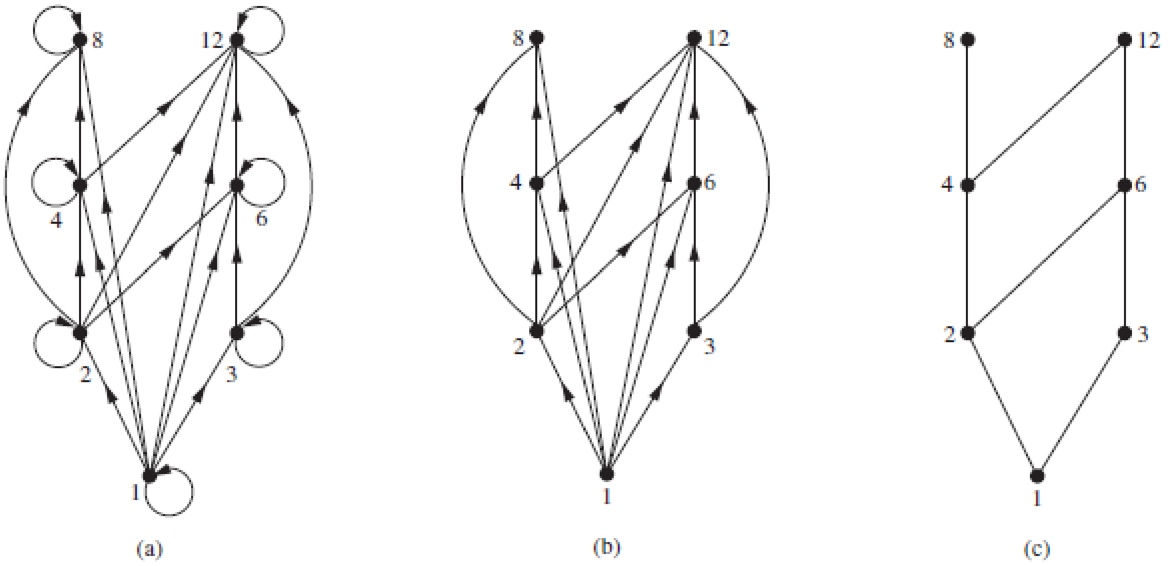


Figure 3 Constructing the Hasse Diagram of ({1, 2, 3, 4, 6, 8, 12}, |).

Maximal and Minimal Elements

- An element of a poset is called maximal if it is not less than any element of the poset. That is, a is maximal in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$.
- Similarly, an element of a poset is called minimal if it is not greater than any element of the poset. That is, a is minimal if there is no element $b \in S$ such that $b \prec a$.
- Maximal and minimal elements are easy to spot using a Hasse diagram. They are the “top” and “bottom” elements in the diagram.

Question1

- Which elements of the poset ({2, 4, 5, 10, 12, 20, 25}, |) are maximal, and which are minimal?
- *Solution:* The Hasse diagram in Figure 4 for this poset shows that the maximal elements are 12, 20, and 25, and the minimal elements are 2 and 5. As this example shows, a poset can have more than one maximal element and more than one minimal element.

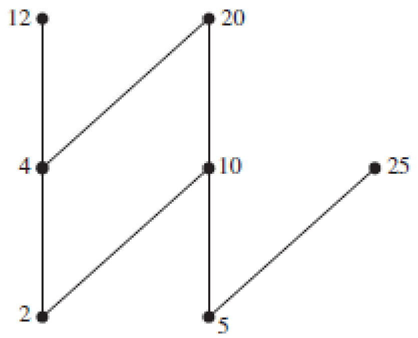


Figure 4 The Hasse Diagram of a poset

Question2

- Determine whether the posets represented by each of the Hasse diagrams in Figure 5 have a greatest element and a least element.
- *Solution:* The least element of the poset with Hasse diagram (a) is *a*. This poset has no greatest element. The Hasse diagram (b) has neither a least nor a greatest element. The Hasse diagram (c) has no least element. Its greatest element is *d*. The poset with Hasse diagram (d) has least element *a* and greatest element *d*.

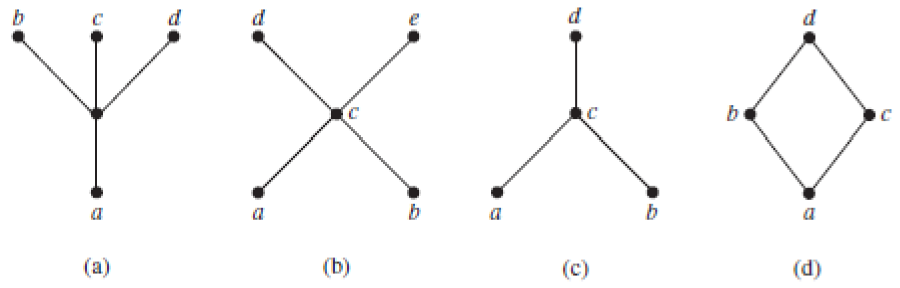


Figure 5 The Hasse Diagram of four posets

Upper and Lower Bounds

- Sometimes it is possible to find an element that is greater than or equal to all the elements in a subset A of a poset $(S, \preccurlyeq)$. If u is an element of S such that $a \preccurlyeq u$ for all elements $a \in A$, then u is called an upper bound of A.

- Likewise, there may be an element less than or equal to all the elements in A. If l is an element of S such that $l \preccurlyeq a$ for all elements $a \in A$, then l is called a lower bound of A.

Example

- Find the lower and upper bounds of the subsets

  $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown in Figure 6.

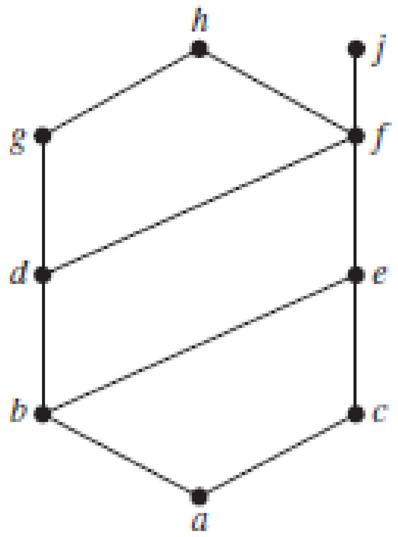


Figure 6 The Hasse Diagram of a posets

Solution

- The upper bounds of $\{a, b, c\}$ are e, f, j, and h, and its only lower bound is a.
- There are no upper bounds of $\{j, h\}$, and its lower bounds are a, b, c, d, e, and f .
- The upper bounds of $\{a, c, d, f\}$ are f , h, and j , and its lower bound is a.

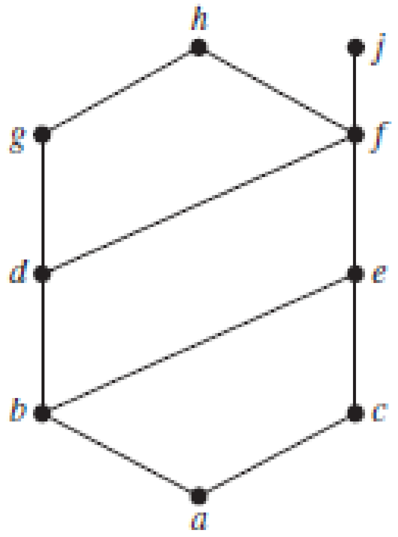


Figure 6 The Hasse Diagram of a posets

Least Upper and Greatest Lower Bounds

- The element x is called the least upper bound of the subset A if x is an upper bound that is less than every other upper bound of A. Because there is only one such element, if it exists, it makes sense to call this element *the* least upper bound. x is the least upper bound of A if a x whenever $a \in A$, and x z whenever z is an upper bound of A.

- Similarly, the element y is called the greatest lower bound of A if y is a lower bound of A and z y whenever z is a lower bound of A.

Question

- Find the greatest lower bound and the least upper bound of $\{b, d, g\}$, if they exist, in the poset shown in Figure 6.
- *Solution:* The upper bounds of $\{b, d, g\}$ are g and h. Because $g \prec h$, g is the least upper bound.

  The lower bounds of $\{b, d, g\}$ are a and b. Because $a \prec b$, b is the greatest lower bound.

Lattices

- A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a lattice. Lattices have many special properties.

- Furthermore, lattices are used in many different applications such as models of information flow and play an important role in Boolean algebra.

Question

Determine whether the posets represented by each of the Hasse diagrams in Figure 7 are lattices.

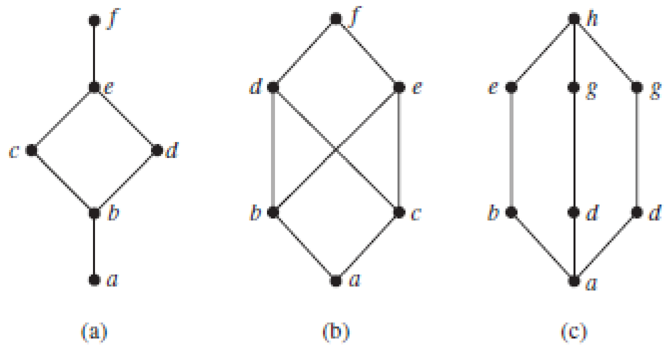


Figure 7 The Hasse Diagram of three posets

Solution

- The posets represented by the Hasse diagrams in (a) and (c) are both lattices because in each poset every pair of elements has both a least upper bound and a greatest lower bound.

- The poset with the Hasse diagram in (b) is not a lattice, because the elements b and c have no least upper bound. To see this, note that each of the elements d, e, and f is an upper bound, but none of these three elements precedes the other two with respect to the ordering of this poset.