# Ceng 124
# Discrete Structures

2018-2019 Spring Semester

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# Topics

- 10.3 Representing Graphs and Graph Isomorphism
- 10.4 Connectivity

# Representing Graphs and Graph Isomorphism

- Sometimes, two graphs have exactly the same form, in the sense that there is a one-to-one correspondence between their vertex sets that preserves edges. In such a case, we say that the two graphs are **isomorphic**.

- Determining whether two graphs are isomorphic is an important problem of graph theory.

# Representing Graphs

- One way to represent a graph without multiple edges is to list all the edges of this graph.
- Another way to represent a graph with no multiple edges is to use adjacency lists, which specify the vertices that are adjacent to each vertex of the graph.

# Example

- Use adjacency lists to describe the simple graph given in Figure 1.

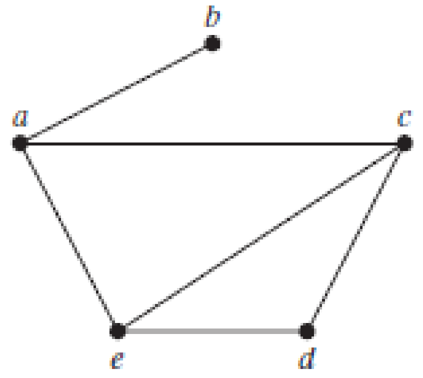


Figure 1: A Simple Graph

**TABLE 1** An Adjacency List for a Simple Graph.

| *Vertex* | *Adjacent Vertices* |
|---|---|
| $a$ | $b, c, e$ |
| $b$ | $a$ |
| $c$ | $a, d, e$ |
| $d$ | $c, e$ |
| $e$ | $a, c, d$ |

Solution: Table 1 lists those vertices adjacent to each of the vertices of the graph.

# Example2

- Represent the directed graph shown in Figure 2 by listing all the vertices that are the terminal vertices of edges starting at each vertex of the graph.

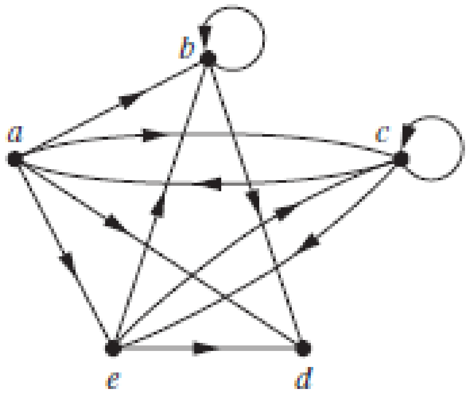


Figure 2 A directed Graph

**TABLE 2** **An Adjacency List for a Directed Graph.**

| *Initial Vertex* | *Terminal Vertices* |
|---|---|
| $a$ | $b, c, d, e$ |
| $b$ | $b, d$ |
| $c$ | $a, c, e$ |
| $d$ | |
| $e$ | $b, c, d$ |

Solution: Table 2 represents the directed graph shown in Figure 2.

# Adjacency Matrices

- Carrying out graph algorithms using the representation of graphs by lists of edges, or by adjacency lists, can be cumbersome if there are many edges in the graph.
- To simplify computation, graphs can be represented using matrices.
- Two types of matrices commonly used to represent graphs will be presented here. One is based on the adjacency of vertices, and the other is based on incidence of vertices and edges.

# Adjacency Matrices (cont.)

- if its adjacency matrix is **A** = [*aij*], then

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

# Example

- Use an adjacency matrix to represent the graph shown in Figure 3.

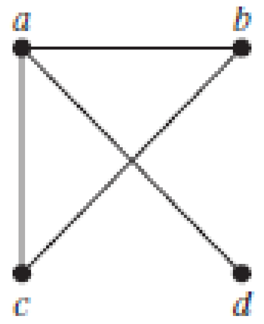


Figure 3: Simple Graph

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Solution: Matrix representaion

# Example2

- Use an adjacency matrix to represent the pseudograph shown in Figure 4.

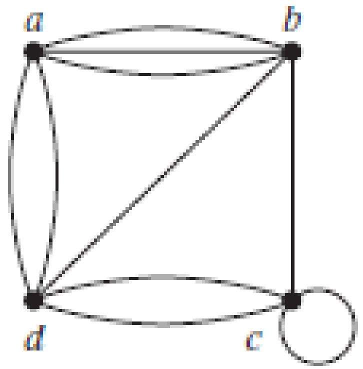


Figure 4: A pseudograph

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Solution: Matrix representation

# Incidence Matrices

- Another common way to represent graphs is to use **incidence matrices**. Let $G = (V, E)$ be an undirected graph. Suppose that $v1, v2, \ldots, vn$ are the vertices and $e1, e2, \ldots, em$ are the edges of $G$. Then the incidence matrix with respect to this ordering of $V$ and $E$ is the $n \times m$ matrix $\mathbf{M} = [mij]$, where $mij = 1$ when edge $ej$ is incident with $vi$, 0 otherwise.

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

# Example

- Represent the graph shown in Figure 5 with an incidence matrix.

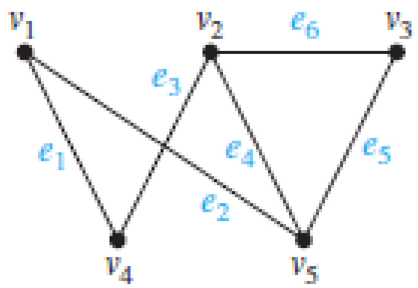


Figure 5: An undirected graph

$$
\begin{array}{c}
 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5
\end{array}
\begin{array}{c}
\begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\
\left[\begin{array}{cccccc}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0
\end{array}\right].
\end{array}
$$

# Example

- Represent the pseudograph shown in Figure 6 using an incidence matrix.

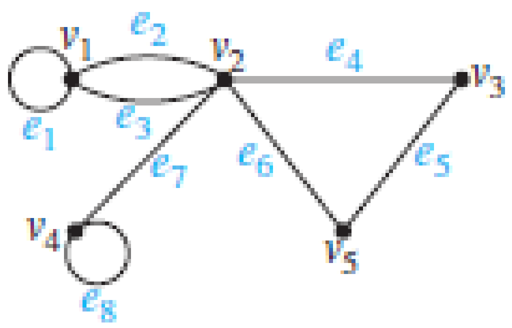


Figure 6: A pseudograph

$$
\begin{array}{c}
\\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}.
\end{array}
$$

# Isomorphism of Graphs

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there exists a one-to-one and onto function $f$ from $V_1$ to $V_2$ with the property that $a$ and $b$ are adjacent in $G_1$ if and only if $f(a)$ and $f(b)$ are adjacent in $G_2$, for all $a$ and $b$ in $V_1$. Such a function $f$ is called an *isomorphism*.* Two simple graphs that are not isomorphic are called *nonisomorphic*.

# Example

- Show that the graphs *G* = *(V ,E)* and *H* = *(W, F)*, displayed in Figure 7, are isomorphic.

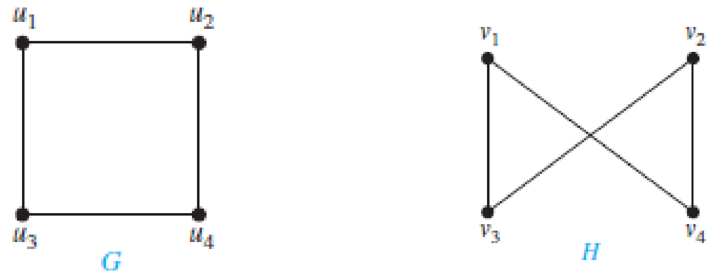


Figure 7: The Graphs G and H

- Solution: The function *f* with *f (u*1*)* = *v*1, *f (u*2*)* = *v*4, *f (u*3*)* = *v*3, and *f (u*4*)* = *v*2 is a oneto-one correspondence between *V* and *W*.

# Determining whether Two Simple Graphs are Isomorphic

- It is often difficult to determine whether two simple graphs are isomorphic. There are $n!$ possible one-to-one correspondences between the vertex sets of two simple graphs with $n$ vertices.
- Isomorphic simple graphs also must have the same number of edges, because the one-to-one correspondence between vertices establishes a one-to-one correspondence between edges. In addition, the degrees of the vertices in isomorphic simple graphs must be the same.

# Question

- Show that the graphs displayed in Figure 8 are not isomorphic.

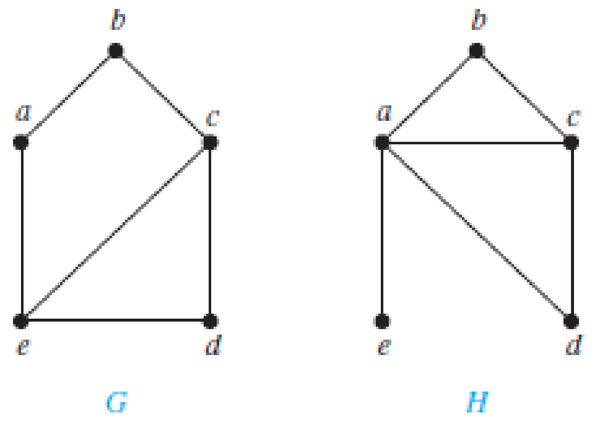


Figure 8: The Graphs G and H

# Solution

- Both $G$ and $H$ have five vertices and six edges. However, $H$ has a vertex of degree one, namely, $e$, whereas $G$ has no vertices of degree one. It follows that $G$ and $H$ are not isomorphic.

# Question2

- Determine whether the graphs shown in Figure 9 are isomorphic.

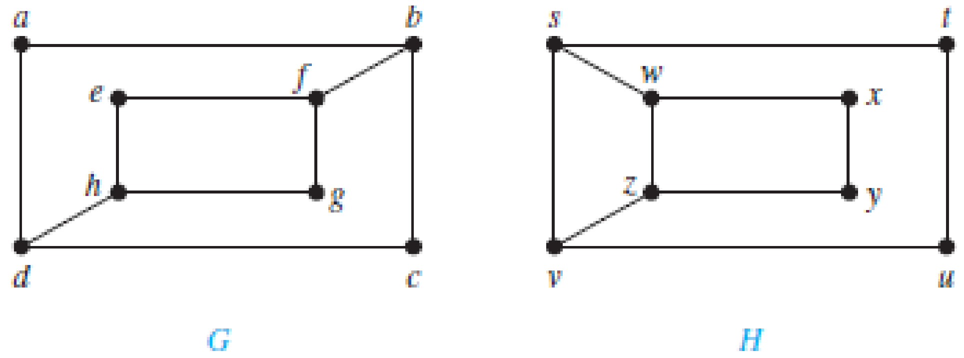


Figure 9: The Graphs G and H

# Solution

- The graphs $G$ and $H$ both have eight vertices and 10 edges. They also both have four vertices of degree two and four of degree three. Because these invariants all agree, it is still conceivable that these graphs are isomorphic.
- However, $G$ and $H$ are not isomorphic. To see this, note that because $\deg(a) = 2$ in $G$, $a$ must correspond to either $t$ , $u$, $x$, or $y$ in $H$, because these are the vertices of degree two in $H$. However, each of these four vertices in $H$ is adjacent to another vertex of degree two in $H$, which is not true for $a$ in $G$.

# Isomorphism by using Adjacency Matrices

- Question: Determine whether the graphs $G$ and $H$ displayed in Figure 10 are isomorphic.

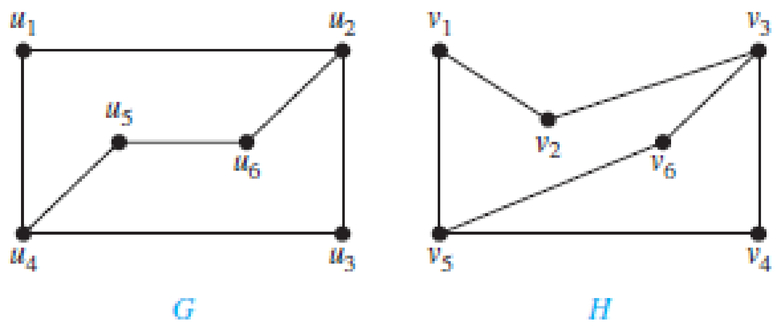


Figure 10: Graphs G and H

# Solution

- Both $G$ and $H$ have six vertices and seven edges. Both have four vertices of degree two and two vertices of degree three. It is also easy to see that the subgraphs of $G$ and $H$ consisting of all vertices of degree two and the edges connecting them are isomorphic. Because $G$ and $H$ agree with respect to these invariants, it is reasonable to try to find an isomorphism $f$.

$$\mathbf{A}_G = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}, \qquad \mathbf{A}_H = \begin{array}{c} \\ v_6 \\ v_3 \\ v_4 \\ v_5 \\ v_1 \\ v_2 \end{array} \begin{array}{c} \begin{array}{cccccc} v_6 & v_3 & v_4 & v_5 & v_1 & v_2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

Because $\mathbf{A}_G = \mathbf{A}_H$, it follows that $f$ preserves edges. We conclude that $f$ is an isomorphism, so $G$ and $H$ are isomorphic.

# Algorithms for Graph Isomorphism

- The best algorithms known for determining whether two graphs are isomorphic have exponential worst-case time complexity. However, linear average-case time complexity algorithms are known that solve this problem, and there is some hope, but also skepticism, that an algorithm with polynomial worst-case time complexity for determining whether two graphs are isomorphic can be found.
- NAUTY software can be downloaded over the Internet and experimented with.

# Applications of Graphs Isomorphism

- Graph isomorphisms, arise in applications of graph theory to chemistry and to the design of electronic circuits, and other areas including bioinformatics and computer vision.
- Modern integrated circuits, known as chips, are miniaturized electronic circuits, often with millions of transistors and connections between them. Because of the complexity of modern chips, automation tools are used to design them.

# 10.4 Connectivity

- A **path** is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph. As the path travels along its edges, it visits the vertices along this path, that is, the endpoints of these edges.

# Example

- In the simple graph shown in Figure 1, $a$, $d$, $c$, $f$ , $e$ is a simple path of length 4, because $\{a, d\}$, $\{d, c\}$, $\{c, f\}$, and $\{f, e\}$ are all edges.

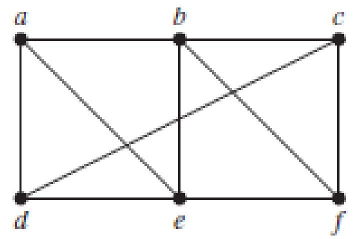


Figure 11: A simple Graph

- However, $d$, $e$, $c$, $a$ is not a path, because $\{e, c\}$ is not an edge. Note that $b$, $c$, $f$ , $e$, $b$ is a circuit of length 4 because $\{b, c\}$, $\{c, f\}$, $\{f, e\}$, and $\{e, b\}$ are edges, and this path begins and ends at $b$. The path $a$, $b$, $e$, $d$, $a$, $b$, which is of length 5, is not simple because it contains the edge $\{a, b\}$ twice.

# Connectedness in Undirected Graphs

An undirected graph is called *connected* if there is a path between every pair of distinct vertices of the graph. An undirected graph that is not *connected* is called *disconnected*. We say that we *disconnect* a graph when we remove vertices or edges, or both, to produce a disconnected subgraph.

Thus, any two computers in the network can communicate if and only if the graph of this network is connected.

# Example

- The graph *G*1 in Figure 12 is connected, because for every pair of distinct vertices there is a path between them.
- However, the graph *G*2 is not connected. For instance, there is no path in *G*2 between vertices *a* and *d*.

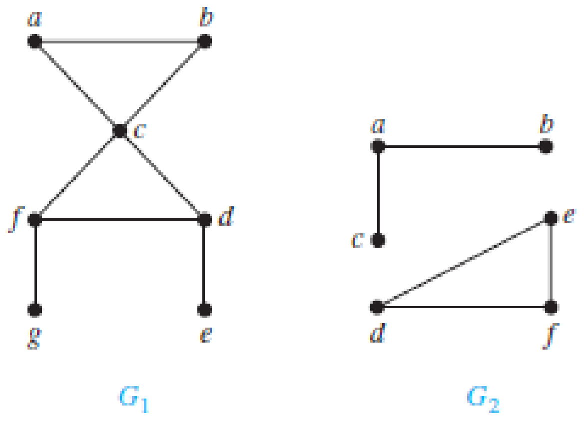


Figure 12: The Graphs G1 and G2

# Paths and Isomorphism

- There are several ways that paths and circuits can help determine whether two graphs are isomorphic. For example, the existence of a simple circuit of a particular length is a useful invariant that can be used to show that two graphs are not isomorphic. In addition, paths can be used to construct mappings that may be isomorphisms.
- A useful isomorphic invariant for simple graphs is the existence of a simple circuit of length $k$, where $k$ is a positive integer greater than 2.

# Question

- Determine whether the graphs *G* and *H* shown in Figure 13 are isomorphic.

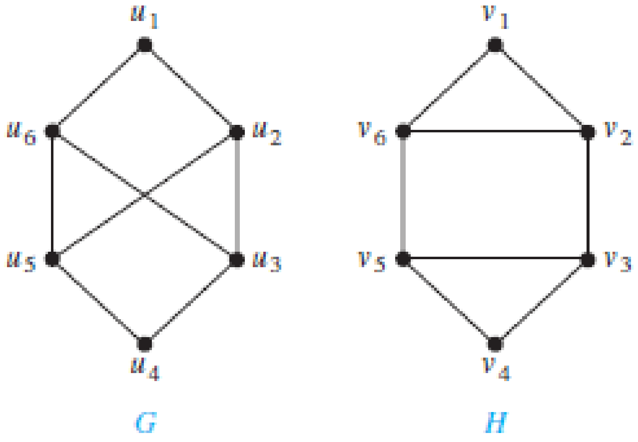


Figure 13: The Graphs G and H

# Solution

- Both $G$ and $H$ have six vertices and eight edges.
- Each has four vertices of degree three, and two vertices of degree two.
- So, the three invariants—number of vertices, number of edges, and degrees of vertices—all agree for the two graphs.
- However, $H$ has a simple circuit of length three, namely, $v1$, $v2$, $v6$, $v1$, whereas $G$ has no simple circuit of length three, as can be determined by inspection (all simple circuits in $G$ have length at least four).
- Because the existence of a simple circuit of length three is an isomorphic invariant, $G$ and $H$ are not isomorphic.

# Question2

- Determine whether the graphs *G* and *H* shown in Figure 14 are isomorphic.

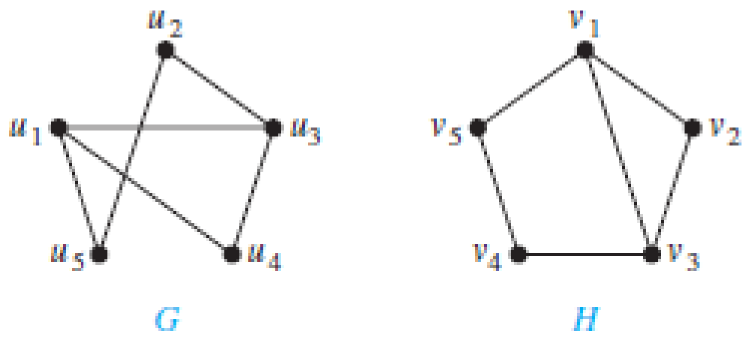


Figure 14: The Graphs G and H

# Solution

- Both $G$ and $H$ have five vertices and six edges,
- both have two vertices of degree three and three vertices of degree two,
- and both have a simple circuit of length three, a simple circuit of length four, and a simple circuit of length five.
- Because all these isomorphic invariants agree, $G$ and $H$ may be isomorphic.
- By following these paths through the graphs, we define the mapping $f$ with $f(u1) = v3$, $f(u4) = v2$, $f(u3) = v1$, $f(u2) = v5$, and $f(u5) = v4$. To show that $f$ is an isomorphism, so $G$ and $H$ are isomorphic, either by showing that $f$ preserves edges or by showing that with the appropriate orderings of vertices the adjacency matrices of $G$ and $H$ are the same.